

# Stack Semantics of Type Theory

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## Abstract

We give a model of dependent type theory with one univalent universe and propositional truncation interpreting a type as a *stack*, generalising the groupoid model of type theory. As an application, we show that countable choice cannot be proved in dependent type theory with one univalent universe and propositional truncation.

## 1 Introduction

The axiom of univalence [15, 14] can be seen as an extension to dependent type theory of the two axioms of extensionality for simple type theory as formulated by Church [2]. This extension is important since, using universe and dependent sums, we get a formal system in which we can represent arbitrary structures (which we can not do in simple type theory) with elegant formal properties. The goal of this paper is to contribute to the meta-theory of such systems by showing that *Markov's principle* and *countable choice* are not provable in dependent type theory extended with one univalent universe and propositional truncation. For simple type theory such independence results can be obtained by using *sheaf semantics*, respectively over Cantor space (for Markov's principle) and open unit interval  $(0, 1)$  (for countable choice). There are however problems with extending sheaf semantics to universes [8, 16]. In order to address these issues we use a suitable formulation of *stack semantics*, which, roughly speaking, replaces *sets* by *groupoids*. The notion of stack was introduced in algebraic geometry [6] precisely in order to solve the same problems that one encounters when trying to extend sheaf semantics to type-theoretic universes. The compatibility condition for gluing local data is now formulated in terms of isomorphisms instead of strict equalities. In this sense, our model can also be seen as an extension of the groupoid model of type theory [7]. One needs to formulate some strict functoriality conditions on the stack gluing operation, which seem necessary to be able to get a model of the required equations of dependent type theory.

We see this work as a first step towards the proof of independence of countable choice from type theory with a hierarchy of univalent universes and propositional truncation, which we hope to obtain by an extension of our model to an  $\infty$ -stack version of cubical type theory [3].

The paper is organized as follows. We first present a slight variation of the groupoid model that we find convenient for expressing the stack semantics. We then explain how to represent propositional truncation in this setting, and how it can be used to formulate countable choice. We then notice that, even in a constructive meta-logic where countable choice fails, the axiom of countable choice does hold in this groupoid model. The groupoid model can be refined rather directly over a Kripke structure, and we present then our notion of stacks over a general topological space together with a proof that we get a model of dependent type theory with one univalent universe and propositional truncation. Instantiating our model to the case of Cantor space and open unit interval  $(0, 1)$  we obtain the results that Markov's principle and countable

$$\begin{array}{c}
\vdash () \qquad \frac{\Gamma \vdash A}{\vdash \Gamma.A} \\
\\
\frac{\vdash \Gamma}{\vdash 1 : \Gamma \rightarrow \Gamma} \qquad \frac{\vdash \tau : \Theta \rightarrow \Delta \quad \vdash \sigma : \Delta \rightarrow \Gamma}{\sigma \tau : \Theta \rightarrow \Gamma} \qquad \frac{\Gamma \vdash A \quad \vdash \sigma : \Delta \rightarrow \Gamma}{\Delta \vdash A\sigma} \qquad \frac{\Gamma \vdash A}{\Gamma.A \vdash \mathbf{q} : A\mathbf{p}} \\
\\
\frac{\Gamma \vdash a : A \quad \vdash \sigma : \Delta \rightarrow \Gamma}{\Delta \vdash a\sigma : A\sigma} \\
\\
A1 = A \qquad A(\sigma\tau) = (A\sigma)\tau \qquad a1 = a \qquad a(\sigma\tau) = (a\sigma)\tau \\
\\
\frac{\Gamma \vdash A}{\vdash \mathbf{p} : \Gamma.A \rightarrow \Gamma} \qquad \frac{\Gamma \vdash A \quad \vdash \sigma : \Delta \rightarrow \Gamma \quad \Delta \vdash a : A\sigma}{\vdash \langle \sigma, a \rangle : \Delta \rightarrow \Gamma.A} \\
\\
1\sigma = \sigma \qquad \sigma 1 = \sigma \qquad \sigma(\tau\nu) = (\sigma\tau)\nu \qquad \mathbf{p}\langle \sigma, a \rangle = \sigma \qquad \mathbf{q}\langle \sigma, a \rangle = a \qquad \langle \mathbf{p}\sigma, \mathbf{q}\sigma \rangle = \sigma \\
\\
\frac{\Gamma.A \vdash B}{\Gamma \vdash \Pi AB} \qquad \frac{\Gamma.A \vdash b : B}{\Gamma \vdash \lambda b : \Pi AB} \qquad \frac{\Gamma \vdash f : \Pi AB \quad \Gamma \vdash a : A}{\Gamma \vdash \mathbf{app}(f, a) : B[a]} \\
\\
\mathbf{app}(\lambda b, a) = b[a] \qquad \lambda \mathbf{app}(f \mathbf{p}, \mathbf{q}) = f
\end{array}$$

Figure 1: Type theory

choice cannot be proved in dependent type theory with one univalent universe and propositional truncation.

## 2 Type theory

As in [1], we will use a generalized algebraic presentation of type theory that is name-free and has explicit substitutions. Building a model is then reduced to defining operations such that certain equations hold. The main rules are presented in figures 1, 2, 3 and 4. We omit equivalence, congruence and substitution rules. The conversion rules assume appropriate typing premises.

We write  $[a]$  for the substitution  $\langle 1, a \rangle$  and  $[a, b]$  for the composite  $[a][b]$ .

## 3 Groupoid model

In this section, we review the *groupoid model* of [7], with a slightly different presentation inspired from [11]. We work in a set theory with a Grothendieck universe  $\mathcal{U}$  (or a suitable constructive version of it if we work in a constructive set theory such as CZF [4]).

A *groupoid* is given by a set  $\Gamma$  of objects and for each  $\rho, \rho' \in \Gamma$  a set  $\Gamma(\rho, \rho')$  of paths/isomorphisms along with a composition operation  $\alpha \cdot \alpha'$  in  $\Gamma(\rho, \rho'')$  for  $\alpha$  in  $\Gamma(\rho, \rho')$  and  $\alpha'$  in  $\Gamma(\rho', \rho'')$  and a unit element  $1_\rho$  in  $\Gamma(\rho, \rho)$  and an inverse operation  $\alpha^{-1}$  in  $\Gamma(\rho', \rho)$  satisfying the usual unit, inverse and associativity laws. We may write  $\alpha : \rho \cong \rho'$  for  $\alpha$  in  $\Gamma(\rho, \rho')$ .

A map  $\sigma : \Delta \rightarrow \Gamma$  between two groupoids  $\Delta$  and  $\Gamma$  is given by a set-theoretic map  $\sigma \nu$  in  $\Gamma$  for  $\nu$  in  $\Delta$  and a map  $\sigma \beta$  in  $\Gamma(\sigma \nu, \sigma \nu')$  for  $\beta$  in  $\Delta(\nu, \nu')$  which commutes with unit, inverse and composition.

$$\begin{array}{c}
\frac{\Gamma \vdash A \text{ small} \quad \Gamma.A \vdash B \text{ small}}{\Gamma \vdash \Pi AB \text{ small}} \qquad \frac{\Gamma.A \vdash B \text{ discrete}}{\Gamma \vdash \Pi AB \text{ discrete}} \\
\\
\Gamma \vdash \mathbf{U} \qquad \frac{\Gamma \vdash A \text{ small discrete}}{\Gamma \vdash |A| : \mathbf{U}} \qquad \frac{\Gamma \vdash a : \mathbf{U}}{\Gamma \vdash \mathbf{El} a \text{ small discrete}} \\
\\
\mathbf{El} |A| = A \qquad \mathbf{|El} a| = a \\
\\
\frac{\Gamma \vdash A \text{ small}}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \text{ discrete}}{\Gamma \vdash A}
\end{array}$$

Figure 2: Universe in type theory

$$\begin{array}{c}
\frac{\Gamma \vdash A \quad \Gamma \vdash a : A \quad \Gamma \vdash b : A}{\Gamma \vdash \text{Path } A a b \text{ discrete}} \qquad \frac{\Gamma \vdash A \text{ small} \quad \Gamma \vdash a : A \quad \Gamma \vdash b : A}{\Gamma \vdash \text{Path } A a b \text{ small}} \\
\\
\frac{\Gamma \vdash A \quad \Gamma \vdash a : A}{\Gamma \vdash \text{refl } a : \text{Path } A a a} \\
\\
\frac{\Gamma \vdash A \quad \Gamma.A.A\mathbf{p}.\text{Path } A\mathbf{p}\mathbf{p}\mathbf{q}\mathbf{p}\mathbf{q} \vdash C \quad \Gamma.A \vdash c : C\langle[\mathbf{q}], \text{refl } \mathbf{q}\rangle \quad \Gamma \vdash a : A \quad \Gamma \vdash b : A \quad \Gamma \vdash p : \text{Path } A a b}{\Gamma \vdash \mathbf{J} c a b p : C[p, b, a]} \\
\\
\mathbf{J} c a a (\text{refl } a) = c[a]
\end{array}$$

Figure 3: Equality in type theory

$$\begin{array}{c}
\frac{\Gamma.A \vdash B}{\Gamma \vdash \Sigma AB} \qquad \frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[a]}{\Gamma \vdash (a, b) : \Sigma AB} \qquad \frac{\Gamma \vdash p : \Sigma AB}{\Gamma \vdash p.1 : A} \qquad \frac{\Gamma \vdash p : \Sigma AB}{\Gamma \vdash p.2 : B[p.1]} \\
\\
(a, b).1 = a \qquad (a, b).2 = b \qquad (p.1, p.2) = p \\
\\
\Gamma \vdash \mathbf{N} \text{ small discrete} \qquad \Gamma \vdash 0 : \mathbf{N} \qquad \frac{\Gamma \vdash n : \mathbf{N}}{\Gamma \vdash \text{succ } n : \mathbf{N}} \\
\\
\frac{\Gamma.\mathbf{N} \vdash C \quad \Gamma \vdash c : C[0] \quad \Gamma.\mathbf{N}.C \vdash d : C[\text{succ } \mathbf{q}]\mathbf{p} \quad \Gamma \vdash n : \mathbf{N}}{\Gamma \vdash \text{rec } c d n : C[n]} \\
\\
\text{rec } c d 0 = c \qquad \text{rec } c d (\text{succ } n) = d\langle [n], \text{rec } c d n \rangle \\
\\
\Gamma \vdash \mathbf{N}_2 \text{ small discrete} \qquad \Gamma \vdash 0 : \mathbf{N}_2 \qquad \Gamma \vdash 1 : \mathbf{N}_2 \\
\\
\frac{\Gamma.\mathbf{N}_2 \vdash C \quad \Gamma \vdash c : C[0] \quad \Gamma \vdash d : C[1] \quad \Gamma \vdash b : \mathbf{N}_2}{\Gamma \vdash \text{rec}_2 c d b : C[b]} \\
\\
\text{rec}_2 c d 0 = c \qquad \text{rec}_2 c d 1 = d
\end{array}$$

Figure 4: Dependent sum, natural numbers and booleans in type theory

A family  $A$  of groupoids indexed over a groupoid  $\Gamma$ , written  $\Gamma \vdash A$ , is given by a family of sets  $A\rho$  for each  $\rho$  in  $\Gamma$  and sets  $A\alpha(u, u')$  for each  $\alpha$  in  $\Gamma(\rho, \rho')$  and  $u \in A\rho$  and  $u' \in A\rho'$ . We may write  $\omega : u \cong_{\alpha} u'$  for  $\omega$  element of  $A\alpha(u, u')$  and we may omit the subscript  $\alpha$  if it is clear from the context. We also have unit  $1_u : u \cong_{1_{\rho}} u$  and inverse  $\omega^{-1} : u' \cong_{\alpha^{-1}} u$  and composition  $\omega \cdot \omega' : u \cong_{\alpha \cdot \alpha'} u''$  also satisfying the unit, inverse and associativity laws. We furthermore should have a *path lifting structure*, which is given by two operations  $u\alpha$  in  $A\rho'$  and  $u \uparrow \alpha : u \cong_{\alpha} u\alpha$  for  $u$  in  $A\rho$  and  $\alpha : \rho \cong \rho'$  satisfying the laws

$$u1_{\rho} = u \quad (u\alpha)\alpha' = u(\alpha \cdot \alpha') \quad u \uparrow 1_{\rho} = 1_u \quad (u \uparrow \alpha) \cdot (u\alpha \uparrow \alpha') = u \uparrow (\alpha \cdot \alpha')$$

We see that  $u \uparrow \alpha$  “lifts” the path  $\alpha : \rho \cong \rho'$  given an initial point  $u$  in  $A\rho$ .

Each  $A\rho$  has a canonical groupoid structure, defining  $A\rho(u, u')$  to be  $A1_{\rho}(u, u')$ . If  $\alpha : \rho \cong \rho'$  we can define a groupoid map  $A\rho \rightarrow A\rho'$  using the lifting operation. We thereby recover the groupoid model as defined in [7].

If  $\sigma : \Delta \rightarrow \Gamma$  and  $\Gamma \vdash A$  we define  $\Delta \vdash A\sigma$  by composition:  $(A\sigma)\nu$  is  $A(\sigma \nu)$  and  $(A\sigma)\beta(v, v')$  is  $A(\sigma \beta)(v, v')$ .

A *section*  $\Gamma \vdash a : A$  is given by a family of objects  $a\rho$  in  $A\rho$  together with a family of paths  $a\alpha : a\rho \cong_{\alpha} a\rho'$  satisfying the laws  $a1_{\rho} = 1_{a\rho}$  and  $a(\alpha \cdot \alpha') = a\alpha \cdot a\alpha'$ .

If  $\Gamma \vdash A$ , we define a new groupoid  $\Gamma.A$ : An object  $(\rho, u)$  in  $\Gamma.A$  is a pair with  $\rho$  in  $\Gamma$  and  $u$  in  $A\rho$  and a path  $(\alpha, \omega) : (\rho, u) \cong (\rho', u')$  is a pair  $\alpha : \rho \cong \rho'$  and  $\omega : u \cong_{\alpha} u'$ . We then have  $\mathbf{p} : \Gamma.A \rightarrow \Gamma$  defined by  $\mathbf{p}(\rho, u) = \rho$  and  $\mathbf{p}(\alpha, \omega) = \alpha$  and the section  $\Gamma.A \vdash \mathbf{q} : A\mathbf{p}$  defined by  $\mathbf{q}(\rho, u) = u$  and  $\mathbf{q}(\alpha, \omega) = \omega$ .

We say that a family  $\Gamma \vdash A$  is *small* if each set  $A\rho$  and  $A\alpha(u, u')$  is in the given Grothendieck universe  $\mathcal{U}$ . We say that this family is *discrete* if the lifting is *uniquely* determined: Given  $u$  in  $A\rho$  and  $\alpha : \rho \cong \rho'$  there is a unique  $u'$  in  $A\rho'$  such that  $A\alpha(u, u')$  is inhabited and this set is a singleton in this case.

We define  $\mathbf{U}$  to be the following groupoid: An object  $X$  in  $\mathbf{U}$  is exactly an element of the given Grothendieck universe  $\mathcal{U}$ , and an element of  $\mathbf{U}(X, X')$  is a bijection between  $X$  and  $X'$ . We can then define the small and discrete family  $\mathbf{U} \vdash \mathbf{El}$  by taking  $\mathbf{El} X$  to be the set  $X$  and  $u \cong_\alpha u'$  to be the subsingleton set  $\{0 \mid u' = \alpha u\}$ , that is  $u \cong_\alpha u'$  is inhabited and is the singleton  $\{0\}$  exactly when  $u' = \alpha u$ .

**Proposition 1.** *The family  $\mathbf{U} \vdash \mathbf{El}$  is a universal small and discrete family: If  $\Gamma \vdash A$  is small and discrete, then there exists a unique map  $|A| : \Gamma \rightarrow \mathbf{U}$  such that  $\mathbf{El} |A| = A$  (with strict equality).*

For  $\Gamma \vdash A$  and  $\Gamma.A \vdash B$  we define  $\Gamma \vdash \Pi AB$  by taking  $(\Pi AB)\rho$  to be the set of functions  $c u$  in  $B(\rho, u)$  and  $c \omega$  in  $B(1_\rho, \omega)(c u, c u')$  commuting with unit and composition, and  $(\Pi AB)\alpha(c, c')$  to be the set of functions  $\gamma \omega : c u \cong_{(\alpha, \omega)} c' u'$  such that  $(\gamma \omega_0) \cdot (c' \beta') = (c \beta) \cdot (\gamma \omega_1)$  if  $\beta : u_0 \cong_\rho u_1$  and  $\beta' : u'_0 \cong_{\rho'} u'_1$  and  $\omega_0 : u_0 \cong_\alpha u'_0$  and  $\omega_1 : u_1 \cong_\alpha u'_1$ . There is then [7, 11] a canonical way to define a composition operation (we need the path lifting structure for  $\Gamma \vdash A$ ) and path lifting structure for  $\Gamma \vdash \Pi AB$ .

**Proposition 2.** *If  $\Gamma.A \vdash B$  is discrete, then so is  $\Gamma \vdash \Pi AB$ .*

If  $\Gamma \vdash A$  and  $\Gamma \vdash a_0 : A$  and  $\Gamma \vdash a_1 : A$  we define the *discrete* family  $\Gamma \vdash C = \mathbf{Path} A a_0 a_1$ . We take  $C\rho$  to be the set  $A1_\rho(a_0\rho, a_1\rho)$  and  $C\alpha(\omega, \omega')$  for  $\alpha : \rho \cong \rho'$  to be the subsingleton  $\{0 \mid \omega \cdot a_1\alpha = a_0\alpha \cdot \omega'\}$ .

It is then possible [7, 11] to check that this defines a model of type theory as presented by the rules of figures 1, 2, 3 and 4.

### 3.1 Propositional truncation

We say that a groupoid is a *proposition* if, and only if, there exists exactly one path between two objects. So  $\Gamma$  is a proposition if, and only if, each set  $\Gamma(\rho, \rho')$  is a singleton.

We define as usual (where names are used for readability)

$$\mathbf{isProp} A = \Pi(x_0 x_1 : A) \mathbf{Path} A x_0 x_1$$

**Proposition 3.** *If  $\Gamma \vdash A$ , then there exists a section  $\Gamma \vdash p : \mathbf{isProp} A$  if, and only if, each groupoid  $A\rho$ ,  $\rho$  in  $\Gamma$ , is a proposition.*

For  $\Gamma \vdash A$  we define  $\Gamma \vdash \|A\|$  as follows. For each  $\rho$  in  $\Gamma$  we take  $\|A\|\rho = A\rho$ , and for each  $\alpha$  in  $\Gamma(\rho, \rho')$ ,  $u$  in  $A\rho$  and  $u'$  in  $A\rho'$  we take  $\|A\|\alpha(u, u')$  to be a fixed singleton  $\{0\}$ . We then have sections of  $\Gamma \vdash \mathbf{isProp} \|A\|$  and  $\Gamma \vdash A \rightarrow \|A\|$ , and given sections of  $\Gamma \vdash \mathbf{isProp} B$  and  $\Gamma \vdash A \rightarrow B$  there is a section of  $\Gamma \vdash \|A\| \rightarrow B$ . In this way, we get a model of the *propositional truncation* operation.

### 3.2 Countable choice

The statement of *countable choice* can be formulated as the type [14]

$$\mathbf{CC} = \Pi(A : \mathbf{N} \rightarrow \mathbf{U})(\Pi(n : \mathbf{N})\|\mathbf{El}(A n)\|) \rightarrow \|\Pi(n : \mathbf{N})\mathbf{El}(A n)\|$$

Notice that we can develop the groupoid model in a constructive meta-theory where countable choice may or may not hold.

**Theorem 1.** *The statement  $\mathbf{CC}$  is valid in the groupoid model (even if countable choice does not hold in the meta-theory).*

*Proof.* It is enough to define  $c A f = f$  and  $c \alpha \omega = 0$  to get  $() \vdash c : \mathbf{CC}$ . □

## 4 Stack model

### 4.1 Groupoid-valued presheaf model

We supposed given a poset with elements  $U, V, W, \dots$ . The groupoid model extends directly as a groupoid-valued presheaf model over this poset. A *context* is now a family of groupoids  $\Gamma(U)$  indexed by element of the given poset such that objects  $\rho$  and paths  $\alpha$  in  $\Gamma(U)$  can be restricted to  $\rho|V$  and  $\alpha|V$  in  $\Gamma(V)$  if  $V \subseteq U$  and the restriction operation defines a groupoid map  $\Gamma(U) \rightarrow \Gamma(V)$ .

For a given context  $\Gamma$ , we define then what is a family  $\Gamma \vdash A$ . It is given by a family of sets  $A\rho$  for each  $U$  and  $\rho$  in  $\Gamma(U)$ , together with a restriction  $u|V$  in  $A(\rho|V)$  for  $u$  in  $A\rho$ , and a family of sets  $A\alpha(u, u')$  with a restriction operation  $\omega|V$  in  $A(\alpha|V)(u|V, u'|V)$  which commutes with unit and composition. Such a family is called *small* if the sets  $A\rho$  and  $A\alpha(u, u')$  are elements in the Grothendieck universe  $\mathcal{U}$ . Furthermore, we should have a lifting operation  $u \uparrow \alpha$  with the law  $(u \uparrow \alpha)|V = (u|V) \uparrow (\alpha|V)$ . A family is called *discrete* if the liftings  $u \uparrow \alpha$  are uniquely determined: given  $U$  and  $\rho$  in  $\Gamma(U)$  and given  $u$  in  $A\rho$  and  $\alpha : \rho \cong \rho'$  there is a unique  $u'$  in  $A\rho'$  such that  $A\alpha(u, u')$  is inhabited, and this set is a singleton in this case.

We can extend the groupoid model to this setting.

An element  $c$  of  $(\Pi AB)\rho$  for  $\rho$  in  $\Gamma(U)$  is a function  $cu$  in  $B(\rho|V, u)$  for  $V \subseteq U$  and  $u$  in  $A(\rho|V)$  and  $c\omega$  in  $B(1_{\rho|V}, \omega)(cu, cu')$  for  $\omega$  in  $A1_{\rho|V}(u, u')$  commuting with unit and composition such that  $(ca)|W = c(a|W)$  and  $(c\omega)|W = c(\omega|W)$  if  $W \subseteq V \subseteq U$ .

An element in  $(\Sigma AB)\rho$  for  $\rho \in \Gamma(U)$  is a pair  $(a, b)$  where  $a \in A\rho$  and  $b \in B(\rho, a)$  with restrictions  $(a, b)|V = (a|V, b|V)$ . Paths in  $(\Sigma AB)\alpha((a, b), (a', b'))$ , where  $\alpha : \rho \cong \rho'$  are pairs  $(\omega, \mu)$  where  $\omega : a \cong_{\alpha} a'$  and  $\mu : b \cong_{(\alpha, \omega)} b'$  with restrictions  $(\omega, \mu)|V = (\omega|V, \mu|V)$ .

### 4.2 Stack structure

We assume given a topological space with a notion of *basic open* closed by intersection and a notion of *covering* of a given basic open by a family of basic open. We consider only covering  $(U_i)_{i \in I}$  of some basic open  $U$  where the set of index  $I$  is *small*. To simplify the presentation we assume that each basic open set is *nonempty*. We write  $U_{ij}$  for  $U_i \cap U_j$  and  $U_{ijk}$  for  $U_i \cap U_j \cap U_k$  when they are non empty.

Since basic opens form a poset, we can consider the notion of type family over this poset as defined in the previous subsection. We are now defining what is a *stack structure* on such a type family  $\Gamma \vdash A$ . For each basic open  $U$  and  $\rho$  in  $\Gamma(U)$  we define first what is the set of *descent data*  $D(A)\rho$ . A *descent datum* is given by a covering  $U_i$  of  $U$  and a family of elements  $u_i$  in  $A\rho|U_i$  and  $\varphi_{ij} : u_i|U_{ij} \cong_{\rho|U_{ij}} u_j|U_{ij}$ , when  $U_i$  meets  $U_j$ , satisfying the *cocycle* conditions<sup>1</sup>

$$\varphi_{ii} = 1_{u_i} \quad \varphi_{ij}|U_{ijk} \cdot \varphi_{jk}|U_{ijk} = \varphi_{ik}|U_{ijk}$$

This forms a set since the index set is restricted to be small (otherwise this might be a proper class in general).

If  $d = (u_i, \varphi_{ij})$  is an element of  $D(A)\rho$  and  $V \subseteq U$  we define its restriction  $d|V$ , element of  $D(A)\rho|V$ , which is the family  $(u_i|V \cap U_i, \varphi_{ij}|V \cap U_{ij})$  restricted to indices  $i$  such that  $V$  meets  $U_i$ . A *gluing operation* **glue**  $d = (u, \varphi_i)$  gives an element  $u$  in  $A\rho$  together with paths  $\varphi_i : u|U_i \cong u_i$  such that  $\varphi_i|U_{ij} \cdot \varphi_j = \varphi_j|U_{ij}$  and satisfies the law **(glue**  $d)|V = \mathbf{glue}(d|V)$ . This means that **glue**  $(d|V)$  should be  $(u|V, \varphi_i|V \cap U_i)$  where we restrict the family to indices  $i$  such that  $V$  meets  $U_i$ .

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<sup>1</sup>The first condition is not logically necessary.

A *stack structure* on a type family  $\Gamma \vdash A$  is given by a gluing operation together with the following sheaf condition: If  $\alpha : \rho \cong \rho'$  is in  $\Gamma(U)$ ,  $u$  and  $u'$  are in  $A\rho$  and  $A\rho'$  respectively and we have a family of paths  $\omega_i : u|_{U_i} \cong_{\alpha|_{U_i}} u'|_{U_i}$  which is compatible (that is  $\omega_i|_{U_{ij}} = \omega_j|_{U_{ij}}$ ), then we have a *unique* path  $\omega : u \cong_{\alpha} u'$  such that  $\omega|_{U_i} = \omega_i$  for all  $i$ <sup>2</sup>.

We recall that a *sheaf*  $F$  is given by a presheaf, i.e. a family of *sets*  $F(U)$  with restriction maps  $u|_V$  in  $F(V)$  for  $V \subseteq U$  such that  $u|_U = u$  and  $(u|_V)|_W = u|_W$  if  $W \subseteq V \subseteq U$ , which satisfies the condition that if we have a covering  $U_i$  of  $U$  and a family of compatible elements  $u_i$  in  $F(U_i)$  (i.e.  $u_i|_{U_{ij}} = u_j|_{U_{ij}}$ ) then there exists a unique  $u$  in  $F(U)$  such that  $u|_{U_i} = u_i$  for all  $i$ . While being a *sheaf* is a *property*, our notion of *stack* is in general a *structure*. (It is a property however when  $\Gamma \vdash A$  is discrete.)

The functoriality property  $(\mathbf{glue} \, d)|_V = \mathbf{glue} \, (d|_V)$  will be crucial for checking that we do get a model of type theory with dependent product. A prime example of a stack which is not a sheaf is the universe of sheaves: If we define  $F(U)$  to be the collection of small sheaves over  $U$  then there is a natural restriction operation  $F(U) \rightarrow F(V)$  for  $V \subseteq U$ , and one can check that the gluing of a compatible family of elements is not unique up to strict equality in general (but it is unique up to isomorphism). Notice that if we try to define the stack structure using global choice as in [6], 3.3.1, page 28, then the functoriality condition will not hold. There is however a more canonical definition of gluing which satisfies this condition, which will provide the interpretation of a univalent universe.

### 4.3 Dependent product

The collection of types with a stack structure is closed under dependent product.

**Theorem 2.** *If  $\Gamma.A \vdash B$  has a stack structure then  $\Gamma \vdash \Pi A B$  has a stack structure.*

*Proof.* Let  $(u_i, \varphi_{ij}) \in D(\Pi A B)\rho$  be a descent datum on a covering  $U_i$  of  $U$ . We construct a glue  $(u, \varphi_i)$  that commutes with restriction.

Given  $x, x' \in A(\rho|_V)$  and  $\nu : x \cong x'$  on  $V \subseteq U$ , we construct  $(u x, \varphi_i x)$  as the glue of  $d_x = (u_i x, \varphi_{ij} x)$  and  $u \nu : u x \cong_{\nu} u x'$  as the unique path matching  $u x \cong u_i x \cong_{\nu|_{V \cap U_i}} u_i x' \cong u x'$  given by the composite of  $\varphi_i x$ ,  $u_i \nu$  and the inverse of  $\varphi_i x'$  on  $V \cap U_i$ . In particular  $V \subseteq U_i$ , then this completely determines  $\varphi_i : u|_{U_i} \cong u_i$ . The uniqueness of  $u \nu$  is needed to show that  $u$  respects units and composites as well as restriction of paths. For  $u$  to also respect restriction of objects we need the fact that  $(\mathbf{glue} \, d_x)|_W = \mathbf{glue} \, d_x|_W = \mathbf{glue} \, d_{x|_W}$  for  $W \subseteq V$ .

Let  $\omega_i : u|_{U_i} \cong_{\alpha|_{U_i}} u'|_{U_i}$  be a matching family of paths. We show that there is a unique glue  $\omega : u \cong_{\alpha} u'$ . It is uniquely determined by the glues  $\omega \nu : u x \cong_{(\alpha|_{V \cap U_i})} u' x'$  of  $\omega_i \nu : u x \cong_{(\alpha|_{V \cap U_i \cap U_i})} u' x'$  for  $\nu : x \cong_{\alpha|_V} x'$ . In particular,  $\omega \nu = \omega_i \nu$  if  $V \subseteq U_i$ . Again, the uniqueness of  $\omega \nu$  lets us show that  $\omega$  respects composites and restrictions.  $\square$

### 4.4 Universe of sheaves

We define  $\mathbf{U}(V)$  to be the collection of all *small* sheaves over  $V$ . There is a natural restriction operation  $\mathbf{U}(V) \rightarrow \mathbf{U}(W)$  if  $W \subseteq V$ .

**Theorem 3.**  *$\mathbf{U}$  has a stack structure.*

*Proof.* Let  $F_i \in \mathbf{U}(U_i)$  with  $\varphi_{ij} : F_i|_{U_{ij}} \cong F_j|_{U_{ij}}$  be a descent datum on a cover  $U_i$  of  $U$ . We construct a glue  $F \in \mathbf{U}(U)$  and  $\varphi_i : F|_{U_i} \cong u_i$ . We define  $F(V)$  for  $V \subseteq U$  as the set of families  $(x_i)_i$  where  $x_i \in F_i(V \cap U_i)$  and  $\varphi_{ij}(x_i) = x_j$ . Furthermore, we define  $F(V) \rightarrow F(W)$  for  $W \subseteq V$

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<sup>2</sup>Similarly we can define the notion of a stack structure on a given presheaf of groupoids. Notice that we shall not require the context  $\Gamma$  to have a stack structure.

component-wise by the restriction  $F_i(V \cap U_i) \rightarrow F_i(W \cap U_i)$  and  $\varphi_i$  by the projection to the  $i$ -th component. For  $\varphi_i$  to be an isomorphism we need the fact  $\varphi_{ii} = 1$  and  $\varphi_{ij} \cdot \varphi_{jk} = \varphi_{ik}$ .

We claim that the presheaf  $F$  satisfies the sheaf property. Indeed, let  $v_k \in u(V'_k)$  be a matching family for  $F$  on a cover  $V'_k$  of  $V$ . The  $i$ -th components of  $v_k$  are a matching family for  $u_i$  on the induced cover  $V \cap U_i \cap V'_k$  of  $V \cap U_i$  and the gluing operation  $D(F_i)(V \cap U_i \cap V'_k) \rightarrow F_i(V \cap U_i)$  of a discrete stack is a bijection so that we obtain a glue  $v \in F(V)$  of  $v_k$  by gluing component-wise. This glue is unique because it is component-wise unique.

Let now  $\omega_i : F|U_i \cong G|U_i$  be a matching family of paths. For  $x \in F(V)$  the family  $\omega_i x$  in  $G(V \cap U_i)$  is compatible because  $\omega_i = \omega_j$  on  $V \cap U_{ij}$ . We define  $\omega x$  to be the unique glue in  $G(V)$  such that  $(\omega x)|V \cap U_i = \omega_i x$ . The uniqueness of glues allows us to verify that  $\omega$  respects restriction and that the such defined  $\omega$  is the unique path that agrees with  $\omega_i$  on  $U_i$ .  $\square$

We define  $\mathbf{U} \vdash \mathbf{El}$  by taking  $\mathbf{El} F$  to be the small set  $F(V)$  if  $F$  is in  $\mathbf{U}(V)$  and  $\mathbf{El} \alpha(a, a')$  to be the set  $\{0 \mid \alpha a = a'\}$  if  $\alpha$  is an isomorphism between  $F$  and  $F'$  in  $\mathbf{U}(V)$  and  $a$  is in  $F(V)$  and  $a'$  is in  $F'(V)$ .

**Theorem 4.** *The family  $\mathbf{U} \vdash \mathbf{El}$  is a universal small and discrete stack: if  $\Gamma \vdash A$  is small and discrete stack, there exists a unique map  $|A| : \Gamma \rightarrow \mathbf{U}$  such that  $\mathbf{El} |A| = A$  (with strict equality).*

## 4.5 Dependent sums

**Theorem 5.** *If  $\Gamma \vdash A$  and  $\Gamma.A \vdash B$  have stack structures then we can glue descent data and paths in  $\Gamma \vdash \Sigma AB$ .*

*Proof.* Let  $((u_i, v_i), (\omega_{ij}, \mu_{ij})) \in D(\Sigma AB)\rho$  be a descent datum on a covering  $U_i$  of  $U$ . We construct a glue  $((u, v), (\omega_i, \mu_i))$  that commutes with restriction.

Let  $(u, \omega_i)$  to be the glue of the datum  $(u_i, \omega_{ij}) \in D(A)\rho$ . We describe a descent datum in  $D(B)(\rho, a)$ . The object part of this descent datum is given by  $v_i \omega_i^{-1} \in B(\rho|U_i, u|U_i)$ . We have then paths  $(v_i \omega_i^{-1} \uparrow \omega_i) : v_i \omega_i^{-1} \cong_{\omega_i} v_i$  and thus paths

$$(v_i \omega_i^{-1} \uparrow \omega_i)|U_{ij} \cdot \mu_{ij} \cdot (v_j \omega_j^{-1} \uparrow \omega_j)^{-1}|U_{ij} : v_i \omega_i^{-1}|U_{ij} \cong v_j \omega_j^{-1}|U_{ij}$$

These satisfy the cocycle condition. Thus we have a descent datum in  $D(B)(\rho, u)$ . Let  $(v, \mu'_i)$  be the glue of this datum. We have paths  $\mu_i = \mu'_i \cdot (v_i \omega_i^{-1} \uparrow \omega_i) : v|U_i \cong_{\omega_i} v_i$

We then take the glue of  $((u_i, v_i), (\omega_{ij}, \mu_{ij}))$  to be given by  $((u, v), (\omega_i, \mu_i))$ . Since

$$\mu'_i|U_{ij} \cdot (v_i \omega_i^{-1} \uparrow \omega_i)|U_{ij} \cdot \mu_{ij} \cdot (v_j \omega_j^{-1} \uparrow \omega_j)^{-1}|U_{ij} = \mu'_j|U_{ij}$$

we have that  $\mu_i|U_{ij} \cdot \mu_{ij} = \mu_j|U_{ij}$ .

Let  $\alpha : \rho \cong \rho'$ . Given a matching family of paths  $(\omega_i, \mu_i) : (u, v)|U_i \cong_{\alpha|U_i} (u', v')$ . Since  $A$  have a stack structure we have a unique  $\omega : u \cong_{\alpha} u'$  with  $\omega|U_i = \omega_i$ . But then  $\mu_i : v|U_i \cong_{(\alpha, \omega)|U_i} v'|U_i$  is a matching family for the stack  $B$  and thus have a unique  $\mu : v \cong_{(\alpha, \omega)} v'$  where  $\mu|U_i = \mu_i$ .  $\square$

## 4.6 Paths

**Proposition 4.** *If  $\Gamma \vdash A$  has a stack structure and  $\Gamma \vdash a_0 : A$  and  $\Gamma \vdash a_1 : A$  then  $\Gamma \vdash \text{Path } A a_0 a_1$  has a discrete stack structure.*



## 5 Countable choice

### 5.1 Propositional truncation in the stack model

If  $\Gamma \vdash A$  and  $\rho$  in  $\Gamma(U)$  we define  $\|A\|_\rho$  to be the proposition on the following *set* of all families  $(U_i, a_i)_{i \in I}$  where  $(U_i)_{i \in I}$  is a covering of  $U$  and  $a_i$  is an element in  $A\rho|U_i$ . Notice that this forms a set since the index set  $I$  is restricted to be small. (Without this restriction, we will get a *class* and not a *set* in general.) The restriction  $(U_i, a_i)|V$  is  $(U_i \cap V, a_i|U_i \cap V)$  where we restrict the indices to  $i$  such that  $U_i$  meets  $V$ . This always has a stack structure: if we have a covering  $(V_l)_{l \in L}$  of  $U$  and for each  $l$  in  $L$ , we have an element  $u_l = (U_{li}, a_{li})_{i \in I_l}$  of  $\|A\|_\rho|V_l$  this family  $u_l$  always defines in a unique way a descent datum and we can consider the family  $(U_{li}, a_{li})_{(l,i) \in J}$  with  $J = \Sigma(l \in L)I_l$  which defines a gluing of the family  $u_l$ . This operation furthermore satisfies the functoriality condition.

### 5.2 A stack model where countable choice does not hold

We write  $U, V, \dots$  non empty open rational intervals included in the open unit interval  $(0, 1)$ . For each  $n$ , and  $i = 1, \dots, n$  we let  $U_i^n$  be  $((i-1)/(n+1), (i+1)/(n+1))$  so that  $U_i^n, i = 1, \dots, n$  is a covering of  $(0, 1)$ .

We let  $|\mathbb{N}|$  be the constant presheaf where each  $|\mathbb{N}|(U)$  is the set  $\mathbb{N}$  of natural numbers and  $\mathbb{N} = \text{El } |\mathbb{N}|$ . We have [13]

**Lemma 1.**  $|\mathbb{N}|$  is a sheaf.

It is also well-known that in the sheaf model over  $(0, 1)$ , there are Dedekind reals that are not Cauchy reals [13]. It is simple to transform this fact to a counter-example to our type-theoretic version of countable choice.

We define  $A : \mathbb{N} \rightarrow \mathbf{U}$  by letting  $An$  be the subsheaf of the constant sheaf  $\mathbb{Q}$

$$(An)(V) = \{r \in \mathbb{Q} \mid \forall(x \in V) |x - r| < \frac{1}{n+1}\}$$

Notice that each  $i/(n+1)$  is an element of  $(An)(U_i^n)$ .

**Proposition 5.** *In this model*

1. the type  $\Pi(n : \mathbb{N})\|\text{El } (An)\|$  is inhabited
2. the type  $\Pi(n : \mathbb{N})\text{El } (An)$ , and hence also the type  $\|\Pi(n : \mathbb{N})\text{El } (An)\|$ , is empty

*Proof.* For each open set  $U$ , we let  $s_U n$  be  $(U \cap U_i^n, i/(n+1))$  in  $\|\text{El } (An)\|(U)$ . Since we have  $(s_U n)|V = s_V n$  if  $V \subseteq U$ , this defines an element of  $\Pi(n : \mathbb{N})\|\text{El } (An)\|$ .

For the second point, it is enough to notice that, for each given  $U$ , the set

$$(An)(U) = \{r \in \mathbb{Q} \mid \forall(x \in U) |x - r| < \frac{1}{n+1}\}$$

is empty for  $n$  large enough. □

**Corollary 1.** *In this model, the principle of countable choice CC does not hold.*

**Corollary 2.** *One cannot show countable choice in type theory with one univalent universe and propositional truncation.*

## 6 Markov's principle

The interpretation of the type  $\mathbf{N}$  was especially simple on the space  $(0, 1)$  using the fact that its basic open are *connected*. We will now consider the “dual” case where the space is *totally disconnected*. We assume from now on that that basic open are non zero element of a boolean algebra with decidable equality. We write  $e, e', \dots$  these basic open. We consider only covering of  $e$  given by a finite partition  $e_i, i \in I$  of  $e$  which is a *finite set* of disjoint elements  $\leq e$ , such that  $e = \bigvee_{i \in I} e_i$ .

Given a type family  $\Gamma \vdash A$  and  $\rho \in \Gamma(e)$ , a descent datum  $d \in D(A)\rho$  for this family is now simply given by a partition  $e_i, i \in I$  of  $e$  and a family  $u_i \in A\rho|e_i$ .

We can now *strengthen* the notion of stack structure by imposing the further condition of *strict gluing condition*: For  $d = (u_i) \in D(A)\rho$  we have  $(\mathbf{glue} d)|e_i = u_i$ . This states that the required equalities between  $(\mathbf{glue} d)|e_i$  and  $u_i$  are *strict* equalities. (This can also be stated as: If  $I$  has exactly one element then  $\mathbf{glue}(u_i) = u_i$ .)

This refinement is needed for the elimination of natural numbers and booleans in the universe.

**Proposition 6.** *If  $\Gamma.A \vdash B$  satisfies the strict gluing condition, then so does  $\Gamma \vdash \Pi A B$ .*

**Proposition 7.** *If  $\Gamma \vdash A$  and  $\Gamma.A \vdash B$  satisfy the strict gluing condition, then so does  $\Gamma \vdash \Sigma A B$ .*

If  $\mathbf{U}(e)$  is the collection of sheaves on  $e$ , we can refine the stack structure on  $\mathbf{U}$  in order to satisfy the strict gluing condition: If  $e_i$  is a partition of  $e$  and  $F_i$  is a sheaf on  $e_i$  we define  $F = \mathbf{glue}(e_i, F_i)$  by taking  $F(e')$ , for  $e' \leq e$ , to be the product of all  $F_i(e' \wedge e_i)$  if  $e'$  meets strictly more than one  $e_i$ , and to be *exactly*  $F_i(e')$  if  $e' \leq e_i$ . This defines a sheaf, and the functoriality law  $\mathbf{glue}(e_i, F_i)|e' = \mathbf{glue}(e_i \wedge e', F_i|e_i \wedge e')$  is satisfied.

### 6.1 Natural numbers and booleans

We define the sheaf  $|\mathbf{N}|$  by taking  $|\mathbf{N}|(e)$  to be the set of families  $(e_i, n_i)$  where  $e_i$  is a partition of  $e$  and  $n_i \neq n_j$  if  $i \neq j$ . We define similarly  $|\mathbf{N}_2|$  where  $n_i$  can only take the values 0 or 1, and  $|\mathbf{N}_1|(e) = \{0\}$ , and  $|\mathbf{N}_0|(e)$  is the empty set. We define then  $\mathbf{N} = \mathbf{El} |\mathbf{N}|$  and  $\mathbf{N}_k = \mathbf{El} |\mathbf{N}_k|$  for  $k = 0, 1, 2$ .

We define  $\text{succ}(e_i, n_i)$  to be  $(e_i, n_i + 1)$  and  $0(e)$  is the element  $(e, 0)$ .

The  $\text{rec}$  operator is then defined as follows: Let  $\rho \in \Gamma(e)$ .

$$\begin{aligned} (\text{rec } c d)\rho 0 &= c\rho \\ (\text{rec } c d)\rho (n + 1) &= d(\rho, n, (\text{rec } c d)\rho n), \text{ where } n \in \mathbf{N} \\ (\text{rec } c d)\rho (e_i, n_i) &= \mathbf{glue}((\text{rec } c d)\rho|e_i n_i) \end{aligned}$$

We remark that the strict gluing condition is needed to make the above definition work, i.e. so that for  $m \in \mathbf{N}(e)$  we have  $((\text{rec } c d)\rho m)|e' = (\text{rec } c d)\rho|e' m|e'$ .

### 6.2 A stack model where Markov's principle does not hold

We can express Markov's principle in type theory by the type:

$$\text{MP} = \Pi(h : \mathbf{N} \rightarrow \mathbf{N}_2)(\neg\neg(\Sigma(x : \mathbf{N})\text{El isZero}(hx)) \rightarrow \Sigma(x : \mathbf{N})\text{El isZero}(hx))$$

where  $\text{isZero} : \mathbf{N}_2 \rightarrow \mathbf{U}$  is defined by  $\text{isZero} = \lambda y. \text{rec}_2 |\mathbf{N}_1| |\mathbf{N}_0| y$ .

We could also consider the version where we use weak existential  $\exists(x : A)B = \|\Sigma(x : A)B\|$  instead of sigma type, but by Exercise 3.19 of [14], the two versions are logically equivalent.

Take a countably infinite set of variables  $p_0, p_1, \dots$ . Consider the free boolean algebra generated by the atomic formulae  $p_n$ . We write  $p_n = 0$  for  $\neg p_n$  and  $p_n = 1$  for  $p_n$ . An object  $e$  in this algebra represents then a compact open in Cantor space  $\{0, 1\}^{\mathbb{N}}$ , where a conjunctive formula  $\bigwedge p_i = b_i$  represents the set of sequences in  $\{0, 1\}^{\mathbb{N}}$  having value  $b_i$  at index  $i$ . A formula  $e$  in the algebra is then a finite disjunction of these.

We have an interpretation of type theory in stacks over this algebra, and we are going to see that Markov's principle is not valid in this interpretation. We define  $\mathbf{f}$  in  $\mathbf{N} \rightarrow \mathbf{N}_2$  by taking  $\mathbf{f} | e \mathbf{n}$  to be  $(e_0, 0), (e_1, 1)$  where  $e_b$  is  $e \wedge (p_n = b)$  if  $e$  meets both  $(p_n = 0)$  and  $(p_n = 1)$  and  $\mathbf{f} | e \mathbf{n}$  to be  $b$  if  $e \leq (p_n = b)$ .

**Proposition 8.** *In this model*

1.  $\neg\neg(\Sigma(x : \mathbf{N}) \text{El isZero}(\mathbf{f} x))$  *is inhabited.*

2.  $\Sigma(x : \mathbf{N}) \text{El isZero}(\mathbf{f} x)$  *is not inhabited.*

*Proof.* To show that  $\neg\neg(\Sigma(x : \mathbf{N}) \text{El isZero}(\mathbf{f} x))$  is inhabited it is sufficient to show that for all  $e$  the set  $(\neg(\Sigma(x : \mathbf{N}) \text{El isZero}(\mathbf{f} x)))(e)$  is empty. For that it will be sufficient to show that for some  $e' \leq e$  we have that  $(\Sigma(x : \mathbf{N}) \text{El isZero}(\mathbf{f} x))(e')$  is not empty. But given any  $e$  we can simply choose  $e' = (p_n = 0) \wedge e$  for some  $n$  big enough. Thus  $(\text{El isZero}(\mathbf{f} n))(e')$  is  $\{0\}$  and  $(\Sigma(x : \mathbf{N}) \text{El isZero}(\mathbf{f} x))(e')$  is not empty.

We now show that  $\Sigma(x : \mathbf{N}) \text{El isZero}(\mathbf{f} x)$  is not inhabited. For any  $n = (e_i, n_i)$  in  $\mathbf{N}(1)$  where  $(e_i)$  is a partition of 1, we can find exactly one  $e_i$  which contains (as a compact open subset of Cantor space) the constant function 1. This element  $e_i$  meets  $p_{n_i} = 1$  and so  $(\text{El isZero}(\mathbf{f} x))(x = n)$  is the empty set.  $\square$

**Corollary 3.** *In this model Markov's principle does not hold.*

**Corollary 4.** *One cannot show Markov's principle in type theory with one univalent universe.*

The situation however is different from the one of countable choice, which provides an alternative argument showing that Markov's principle cannot be proved in type theory with one univalent universe<sup>3</sup>.

**Proposition 9.** *Markov's principle does not hold in the groupoid model in a set theory where Markov's principle does not hold (for instance in suitable sheaf models of CZF [5]).*

## 7 Conclusion

One special case of sheaf models are Boolean-valued models, for instance as in the work [12], and it would be interesting to formulate a stack version of these models as well.

We expect that essentially the same kind of models can be defined over a *site* and not only over a topological space. In particular, it should be possible to extend the sheaf model in [9] to a stack model of type theory with an algebraic closure of a given field, where existence of roots is formulated using propositional truncation (as explained in the cited work, this existence cannot be stated using strong existence expressed by sigma types). Another example could be a stack version of Schanuel topos used in the theory of nominal sets [10].

As stated in the introduction, the argument should generalize to an  $\infty$ -stack version of the cubical set model [3]. The coherence condition on descent data will be infinitary in general, but it will become finitary when we restrict the homotopy level (and empty in particular in the case of propositions).

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<sup>3</sup>This argument gives also a proof that Markov's principle is independent of a hierarchy of univalent universes by considering the cubical set model [3] in a set theory where Markov's principle does not hold.

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